A d.c. $C^1$ function need not be difference of convex $C^1$ functions

DAVID PAVLICA

Abstract. In [2] a delta convex function on $\mathbb{R}^2$ is constructed which is strictly differentiable at 0 but it is not representable as a difference of two convex function of this property. We improve this result by constructing a delta convex function of class $C^1(\mathbb{R}^2)$ which cannot be represented as a difference of two convex functions differentiable at 0. Further we give an example of a delta convex function differentiable everywhere which is not strictly differentiable at 0.

Keywords: differentiability, delta-convex functions

Classification: Primary 26B25; Secondary 26B05

Let $X$ be a normed vector space. We say that a function $f : X \to \mathbb{R}$ is delta convex (d.c.) if there exist continuous convex functions $f_1,f_2$ on $X$ such that $f = f_1 - f_2$.

We denote $B(a,r) = \{ x \in X : \| x - a \| \leq r \}$. Let $g$ be a function defined on an open set $A \subset X$. We say that $L \in X^*$ is the strict derivative at a point $a \in A$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x,y \in B(a,\delta)$ we have

$$|g(x) - g(y) - L(x - y)| \leq \varepsilon \| x - y \|.$$

Note that if a convex function on $X$ is Fréchet differentiable at a point $a$ then it is strictly differentiable at $a$ ([6, Proposition 3.8]).

If $X$ is a finite dimensional space then every function $f \in C^2(X)$ can be represented as $f = f_1 - f_2$, where $f_1,f_2$ are convex and $f_1 \in C^2(X), f_2 \in C^\infty(X)$ (see [3], where other related results are obtained).

In [2], a d.c. function $f : \mathbb{R}^2 \to \mathbb{R}$ is constructed which is strictly differentiable at 0 and is not representable as a difference of two convex functions with this property. But this function is not differentiable everywhere. We shall improve the construction of [2] to obtain a d.c. function of class $C^1(\mathbb{R}^2)$ not representable as a difference of convex functions differentiable at 0.

We shall denote $\lambda_n$ the Lebesgue measure on $\mathbb{R}^n$. We say that $f : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz with the constant $L$ if for each $x, y \in \mathbb{R}^2$ is $|f(x) - f(y)| \leq L \| x - y \|$.

In the following we shall use the notion of the dual convex function.

The author was supported by the grant GAČR 201/03/0931.
Definition. Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a convex function. The dual function \( f^* \) of the function \( f \) is defined on \((\mathbb{R}^n)^*\) by

\[
f^*(x^*) = \sup_{x \in \mathbb{R}^n} \langle x, x^* \rangle - f(x), \quad x^* \in (\mathbb{R}^n)^*.
\]

It follows immediately from the definition that if \( f, g: \mathbb{R}^n \to \mathbb{R} \) are convex functions, \( f \leq g \) and \( f^* \) is finite everywhere then \( g^* \) is finite everywhere. Therefore if \( f \geq \| \cdot \|^2 - 1 \) then \( f^* \) is finite everywhere.

As usual, we identify the dual space \((\mathbb{R}^n)^*\) with \( \mathbb{R}^n \) and \( \langle \cdot, \cdot \rangle \) denotes both the duality and the scalar product.

Facts. If \( f: \mathbb{R}^n \to \mathbb{R} \) is a convex function and \( f^* \) is finite everywhere then

\[
\begin{align*}
(1) & \quad (f^*)^* = f, \\
(2) & \quad x^* \in \partial f(x) \iff x \in \partial f^*(x^*).
\end{align*}
\]

The statement (1) can be found in [4, Theorem 12.2] and (2) in [4, Theorem 23.5].

In [2] a function \( \bar{G}: \mathbb{R}^2 \to \mathbb{R} \) is constructed in the following way.

Fix a sequence of positive integers \( \{k_i\} \) such that \( \cos\left(\frac{2\pi k_i}{k_i}\right) \geq 1 - 2^{-i-3} \) for \( i \in \mathbb{N} \). Let us denote

\[
M := \left\{ \left(2^{-i} \cos\left(\frac{2\pi k_i}{k_i}\right), 2^{-i} \sin\left(\frac{2\pi k_i}{k_i}\right)\right) : i \in \mathbb{N}, k \in \{1, \ldots, k_i\} \right\}.
\]

Set

\[
F(x) = \|x\| + 4\|x\|^2 \quad \text{for } x \in \mathbb{R}^2.
\]

For each \( z \in M \) define

\[
G_z(x) = F(z) + \langle F'(z), x - z \rangle = (8\|z\| + 1)\frac{\langle x, z \rangle}{\|z\|} - 4\|z\|^2.
\]

Since \( F \) is convex we have \( G_z \leq F \) on \( \mathbb{R}^2 \). Let us define for \( x \in \mathbb{R}^2 \)

\[
\bar{G}(x) = \sup \{ G_z(x) : z \in M \}, \quad G(x) = \max\{ \bar{G}(x), \|x\|^2 - 1 \}.
\]

Obviously \( \bar{G} \) and \( G \) are convex functions.

The following 3 lemmas are proved in [2] (Lemmas 3,4,5).

Lemma 1. The function \( \bar{G} \) satisfies

\[
\|x\| + \|x\|^2 \leq \bar{G}(x) \leq \|x\| + 4\|x\|^2 = F(x)
\]

for \( \|x\| < 1 \).
Corollary 1. Therefore $G \equiv \tilde{G}$ on $B(0, 1)$ and $\partial G(0) = \partial \tilde{G}(0) = B(0, 1)$. (Indeed, $\partial(\| \cdot + a\| \cdot \| \cdot \|)^2(0) = B(0, 1)$ for each $a \geq 0$.)

Lemma 2. If $x \in \mathbb{R}^2$, $\|x\| < 1$, $z \in M$, $\|z\| \leq \frac{\|x\|}{9}$ then

$$G_z(x) \leq \tilde{G}(x) - \frac{\|x\|^2}{9}.$$ 

Lemma 3. If $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$ and

$$M_x := \{z \in M : \|z\| \leq 2\|x\|, \langle x, z \rangle \geq \|z\| \cdot \|x\|(1 - 8\|z\|)\}$$

then

$$\tilde{G}(x) = \sup \{G_z(x) : z \in M_x\}.$$ 

Corollary 2. Let $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$. Then there exists a neighbourhood $W$ of $x$ such that, for $w \in W$,

$$G(w) = \sup \{G_z(w) : z \in M_x\}$$

holds.

Proof: The set $N_z := \{u \in \mathbb{R}^2 : 0 < \|u\| < \frac{1}{16}, z \notin M_u\}$ is obviously open for all $z \in M$. Hence

$$U := \bigcap_{z \in M \setminus (M_x \cup B(0, \|x\|/16))} N_z$$

is a neighbourhood of $x$. Since $M_x \cup B(0, \|x\|/16) \supset M_w$ for every $w \in U$, we conclude, using Lemma 2 and Lemma 3 for $w$, that we can put $W = U \cap B(x, \|x\|/2)$. 

Lemma 4. Let $\hat{G}_\alpha : \mathbb{R}^2 \to \mathbb{R}$, $\alpha \in A$, be a family of affine functions with the Lipschitz constant $L$ and $\hat{G}(w) = \sup \{\hat{G}_\alpha(w) : \alpha \in A\}$ for $w \in \mathbb{R}^2$, $\hat{G} : \mathbb{R}^2 \to \mathbb{R}$. Let $x \in \mathbb{R}^2$ and $u^* \in \partial \hat{G}(x)$. Then $\|u^*\| \leq L$.

Proof: The function $\hat{G}(w)$ is obviously Lipschitz with the constant $L$. Therefore $\|u^*\| \leq L$. 

Lemma 5. If $x \in \mathbb{R}^2$, $0 < \|x\| < \frac{1}{16}$ and $x^* \in \partial G(x)$, then

$$\left\| x^* - \frac{x}{\|x\|} \right\| \leq 24\|x\|^{1/2}.$$ 

Proof: Let $z \in M_x$ and

$$y^* = \frac{z}{\|z\|} + 8z \in \partial G_z(x).$$
Clearly
\[
\left\| \frac{z}{\|z\|} - \frac{x}{\|x\|} \right\|^2 = 2 - \frac{2(z, x)}{\|z\|\|x\|}
\]
and by the definition of $M_x$ we have $1 - \frac{(z, x)}{\|z\|\|x\|} \leq 8\|z\|$ and $\|z\| \leq 2\|x\|$. Therefore
\[
\left\| y^* - \frac{x}{\|x\|} \right\| \leq 8\|z\| + \left( \frac{2(z, x)}{\|z\|\|x\|} \right)^{1/2} \leq 16\|x\| + (2 \cdot 8\|z\|)^{1/2}
\]
\[
\leq 16\|x\|^{1/2} + (32\|x\|)^{1/2} \leq 24\|x\|^{1/2}.
\]
Therefore $G_z - \langle \frac{x}{\|x\|}, \cdot \rangle$ is Lipschitz with the constant $24\|x\|^{1/2}$ for $z \in M_x$. Using Corollary 2 and Lemma 4 applied for $G_z - \langle \frac{x}{\|x\|}, \cdot \rangle$, $z \in M_x$, we obtain $\|u^*\| \leq 24\|x\|^{1/2}$ for $u^* \in \partial(G - \langle \frac{x}{\|x\|}, \cdot \rangle)(x)$. Since
\[
x^* - \frac{x}{\|x\|} \in \partial \left( G - \langle \frac{x}{\|x\|}, \cdot \rangle \right)(x),
\]
whenever $x^* \in \partial G(x)$, this completes the proof of Lemma 5.  

By Corollary 1, $G^* \equiv 0$ on $B(0, 1)$ since $G(0) = 0$. Define a function $\alpha: [0, +\infty) \to \mathbb{R}$,
\[
\alpha(t) = 0, \quad t \in [0, 1),
\]
\[
= (t - 1)^4, \quad t \in [1, +\infty),
\]
and $\psi(x^*) := \alpha(\|x^*\|)$, for $x^* \in \mathbb{R}^2$. Then $\psi$ is a convex function on $\mathbb{R}^2$, since $\|\cdot\|$ is convex and $\alpha$ is convex and increasing. Notice that
\[
\psi'(x^*) = 4(\|x^*\| - 1)^3 \frac{x^*}{\|x^*\|}
\]
for $\|x^*\| \geq 1$.

Set $K := G^* + \psi$ and $\tilde{G} := K^*$.

The function $\tilde{G}$ is differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Otherwise there exist $x \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $x^*, y^* \in \partial \tilde{G}(x)$, $x^* \neq y^*$. Then $x \in \partial K(x^*) \cap \partial K(y^*)$.

It is easy to see that then $K$ is affine on $\text{conv}\{x^*, y^*\}$ and $x \in \partial K(z^*)$, for each $z^* \in \text{conv}\{x^*, y^*\}$. Since $K \equiv 0$ on $B(0, 1)$ and $x \neq 0$, the interior of $B(0, 1)$ is disjoint with $\text{conv}\{x^*, y^*\}$. Further there is no line segment in $\partial B(0, 1)$, consequently the function $K$ is affine on some line segment in $\mathbb{R}^2 \setminus B(0, 1)$. Also $\psi$ is affine on this line segment (since $\psi$ and $G^*$ are convex). But it is impossible since $\psi'$ is one-to-one on $\mathbb{R}^2 \setminus B(0, 1)$.  

D. Pavlica
Lemma 6. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in \mathbb{R}^2$, $0 < \|x\| < \delta$, then
\[
\left\| (\tilde{G})'(x) - \frac{x}{\|x\|} \right\| \leq \varepsilon.
\]

Proof: Set
\[
\delta := \min \left\{ \left( \frac{\varepsilon}{9 \cdot 24^3} \right)^2, \frac{1}{16} \right\}.
\]

Let $0 < \|x\| < \delta$. Denote $x^* := (\tilde{G})'(x)$ and $x' := x - \psi'(x^*)$. Then, by Fact (2), $x \in \partial K(x^*)$ and therefore, since $K \equiv 0$ on $B(0, 1)$, we have $\|x^*\| \geq 1$.

Clearly $x' \in \partial (K - \psi)(x^*) = \partial G(x^*)$ and, using again Fact (2), $x^* \in \partial G(x')$.

Further $x' \neq 0$, since if $\|x^*\| = 1$ then clearly $x' = x$ and if $\|x^*\| > 1$ we use $x^* \in \partial G(x')$ and Corollary 1.

Since $\partial G$ is monotone, $0 \in \partial G(0)$ and $x^* \in \partial G(x')$, we have $\langle x', x^* \rangle \geq 0$.

Hence
\[
\langle x', \psi'(x^*) \rangle = \langle x', x^* \rangle \frac{4(\|x^*\| - 1)^3}{\|x^*\|} \geq 0.
\]

Consequently $\|x'\|^2 = \langle x', x - \psi'(x^*) \rangle \leq \langle x', x \rangle \leq \|x'\| \cdot \|x\|$ which implies $\|x'\| \leq \|x\| < \delta$. Now we compute, using Lemma 5 for $x'$,
\[
\left\| (\tilde{G})'(x) - \frac{x}{\|x\|} \right\| \leq \left\| x^* - \frac{x'}{\|x'\|} \right\| + \left\| \frac{x'}{\|x'\|} - \frac{x}{\|x\|} \right\|
\leq 24\|x'\|^{1/2} + \left\| \left( \|x\| \cdot \|x'\| - \|x'\| \cdot \|x\| \right) + \left( \|x\| \cdot \|x'\| - \|x'\| \cdot \|x\| \right) \right\|
\leq 24\|x'\|^{1/2} + \frac{2\|x - x'\|}{\|x\|} \leq 24\|x'\|^{1/2} + \frac{2\|\psi'(x^*)\|}{\|x\|}
\leq 24\delta^{1/2} + \frac{8(\|x^*\| - 1)^3}{\|x\|} \leq 24\delta^{1/2} + 8\cdot 24^3 \|x\|^{3/2}
\leq \delta^{1/2}(24 + 8 \cdot 24^3) \leq \varepsilon,
\]

since by Lemma 5 we also have $\|x^*\| - 1 \leq 24\|x'\|^{1/2}$. \hfill \box

Theorem. The function $H := \tilde{G} - \| \cdot \|$ is a $C^1$ delta-convex function on $\mathbb{R}^2$ and there does not exists a convex function $h$ differentiable at the origin such that $H + h$ is convex.

Proof: As was already proved, $\tilde{G}$ is differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and therefore, since it is convex, $\tilde{G}$ is also $C^1$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Obviously $\| \cdot \|$ is $C^1$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Hence $H \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$. The Fréchet derivative of $H$ at the origin is 0 since, by Lemma 6, for every $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
\|H(u) - H(0)\| = \left| \int_0^1 \langle u, H'(tu) \rangle \, dt \right| \leq \int_0^1 \|H'(tu)\| \, dt \|u\| \leq \varepsilon \|u\|,
\]
for each \( u \in \mathbb{R}^2 \), \( 0 < \|u\| < \delta \). It also follows immediately from Lemma 6 that \( H' \) is continuous at the origin.

Now we shall prove that \( H \) has no control function differentiable at 0. For a contradiction let us suppose that \( h, h + H \) are convex functions on \( \mathbb{R}^2 \) and \( h \) is differentiable at 0. We may assume \( h'(0) = 0 \). Then 0 is the strict derivative of \( h \) at 0 ([3, Proposition 3.8]). Find \( 0 < R < 1/(8^2 \cdot 24^6) \) such that

\[
|h(x) - h(y)| < \frac{1}{48} \|x - y\| \quad \text{if} \quad x, y \in B(0, 2R).
\]

Denote for \( z \in M \)

\[
S_z := \{ x \in [-R/2, R/2]^2 : G(x) = G_z(x) \},
\]

\[
\hat{S}_z := S_z + \psi'(F'(z)), \quad \hat{S} := \bigcup_{z \in M} \hat{S}_z.
\]

**Claim 1.** The function \( \hat{G} \) is affine on \( \hat{S}_z \) for each \( z \in M \). Further, for \( z_1, z_2 \in M \), \( z_1 \neq z_2 \), we have \( \text{int} \hat{S}_{z_1} \cap \text{int} \hat{S}_{z_2} = \emptyset \).

**Proof of Claim 1:**

If \( z \in M \) and \( u \in S_z \) then clearly \( F'(z) \in \partial G(u) \). By Fact (2) we have \( u \in \partial G^*(F'(z)) \). Hence \( u + \psi'(F'(z)) \in \partial K(F'(z)) \). Now, again by Fact (2), \( F'(z) \in \partial \hat{G}(u + \psi'(F'(z))) \). Therefore \( \hat{G} \) is affine on \( \hat{S}_z \).

Finally \( \text{int} \hat{S}_{z_1} \cap \text{int} \hat{S}_{z_2} = \emptyset \) since \( F'(z_1) \neq F'(z_2) \), for \( z_1 \neq z_2 \). \( \square \)

**Claim 2.** \( \hat{S}_z \subset [-R, R]^2 \) for \( z \in M \).

**Proof of Claim 2:**

Let \( z \in M \), \( u \in S_z \). By Lemma 5, since \( F'(z) \in \partial G(u) \), we have \( \|F'(z)\| - 1 \leq 24 \|u\|^{1/2} \leq 24 \cdot (R)^{1/2} \).

We easily compute

\[
\|F'(z)\| = \left\| \frac{z}{\|z\|} + 8z \right\| = 1 + 8 \|z\| > 1.
\]

Hence

\[
\|\psi'(F'(z))\| = \left\| 4(\|F'(z)\| - 1)^3 \cdot \frac{F'(z)}{\|F'(z)\|} \right\| \leq 4 \cdot 24^3 \cdot (R)^{3/2}
\]

\[
< 4 \cdot 24^3 \left( \frac{1}{8^2 \cdot 24^6} \right)^{1/2} R = \frac{R}{2}.
\]

This proves Claim 2. \( \square \)
According to Lemma 2, for each $0 < \delta < 1$, $G = \sup\{G_z : z \in M \setminus B(0, \delta/9)\}$ on $B(0, 1) \setminus B(0, \delta)$.

Hence, for each $\delta > 0$, the function $G$ is defined on $B(0, 1) \setminus B(0, \delta)$ as a supremum of finitely many $G_z$. Therefore $\bigcup_{z \in M} S_z = [-R/2, R/2] \setminus \{(0, 0)\}$. Since $S_z$ are convex we get by Claim 1

$$\lambda_2(\hat{S}) = \sum_{z \in M} \lambda_2(S_z) = R^2.$$  

Without loss of generality we may assume

$$\lambda_2(\hat{S} \cap \{(t_1, t_2) \in \mathbb{R}^2 : 0 \leq t_1 \leq R, -t_1 \leq t_2 \leq t_1\}) \geq \frac{R^2}{4}.$$  

By Fubini’s Theorem

$$\int_0^R \lambda_1(\{t_2 \in [-t_1, t_1] : (t_1, t_2) \in \hat{S}\}) dt_1 \geq \frac{R^2}{4}.$$  

Thus there exists $0 < r < R$ such that

$$\lambda_1(\{t_2 \in [-r, r] : (r, t_2) \in \hat{S}\}) \geq \frac{R}{4} > \frac{r}{4}.$$  

Let us denote for $t \in [-r, r]$

$$\phi(t) := \|(r, t)\|,$$
$$\gamma(t) := \tilde{G}((r, t)),$$
$$\kappa(t) := h((r, t)).$$

By Claim 1 the function $\gamma$ is affine on the interval $\bar{S}_z := \{t \in [-r, r] : (r, t) \in \hat{S}_z\}$ for $z \in M$ and $\lambda_1(\bigcup_{z \in M} \bar{S}_z) \geq r/4$. Therefore there exist $-r \leq s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k \leq r$, $k \in \mathbb{N}$, such that $\gamma$ is affine on $[s_i, t_i]$, for every $1 \leq i \leq k$, and $\sum_{i=1}^k (t_i - s_i) \geq r/5$.

Since $\kappa + \gamma - \phi$ is convex on $[-r, r]$, for each $i = 1, \ldots, k$

$$\kappa'_-(t_i) - \kappa'_+(s_i) + \gamma'_-(t_i) - \gamma'_+(s_i) - \phi'(t_i) + \phi'(s_i) \geq 0$$

holds. Obviously $\gamma'_-(t_i) = \gamma'_+(s_i)$, $i = 1, \ldots, k$.

Hence, by convexity of $\kappa$, we have $\kappa'_-(r) - \kappa'_+(r) \geq \sum_{i=1}^k (\kappa'_-(t_i) - \kappa'_+(s_i)) \geq \sum_{i=1}^k (\phi'(t_i) - \phi'(s_i))$. Since $\kappa$ is Lipschitz with the constant $1/48$ on $[-r, r]$, we have

$$|\kappa'_-(r)| \leq \frac{1}{48}, \quad |\kappa'_+(r)| \leq \frac{1}{48}.$$
By the Mean Value Theorem there exist $\xi_i \in ]s_i, t_i[$ such that $\phi'(t_i) - \phi'(s_i) = \phi''(\xi_i)(t_i - s_i), i = 1, \ldots, k$.

$$
\phi''(\xi_i) = \frac{(r^2 + \xi_i^2)^{1/2} - \frac{\xi_i^2}{(r^2 + \xi_i^2)^{1/2}}}{r^2 + \xi_i^2} = \frac{r^2}{(r^2 + \xi_i^2)^{3/2}} \geq \frac{r^2}{(2r^2)^{3/2}} \geq \frac{1}{4r}.
$$

Finally we obtain

$$
\frac{1}{24} \geq \kappa'_-(r) - \kappa'_+(r) \geq \sum_{i=1}^{k} (\phi'(t_i) - \phi'(s_i)) \geq \frac{1}{4r} \sum_{i=1}^{k} (t_i - s_i) \geq \frac{1}{20},
$$

a contradiction.

If a convex function on a Hilbert space is Fréchet differentiable at some point then it is strictly differentiable at this point. For d.c. functions this need not be true. First example (on $\mathbb{R}^2$) of this phenomenon is probably due to A. Shapiro (see [5], [1] or [6]). But none of these functions is differentiable everywhere.

We shall give an example of a d.c. function on $\mathbb{R}^2$ differentiable at 0 which is of class $C^1$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but is not strictly differentiable at 0.

Set for $(x, y) \in \mathbb{R}^2$

$$
f_1(x, y) = \begin{cases} 
  y & \text{for } y \geq x^2, \\
  x^2 + \frac{y^2}{x^2} - y & \text{for } x^2 > y > 0, \\
  x^2 - y & \text{for } 0 \geq y.
\end{cases}
$$

It is easy to check that $f_1$ is a continuous function with a continuous derivative on $\mathbb{R}^2 \setminus \{(0, 0)\}$. The Hess’s matrix of $y, x^2 + \frac{y^2}{x^2} - y$ and $x^2 - y$ is nonnegative definite for $y > x^2$, for $x^2 > y > 0$ and for $0 > y$, respectively. Since the function $f_1$ has a supporting affine functional at 0 and $f_1$ is differentiable at the points of the sets $\{y = x^2, x \neq 0\}$ and $\{y = 0, x \neq 0\}$, the function $f_1$ is convex on every line, therefore it is convex.

Analogously we prove that

$$
f_2(x, y) = \begin{cases} 
  x^2 + y & \text{for } y \geq 0, \\
  x^2 + \frac{y^2}{x^2} + y & \text{for } 0 > y > -x^2, \\
  -y & \text{for } -x^2 \geq y.
\end{cases}
$$
is a convex function with continuous derivative on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

It is easy to prove that for $(x, y) \in \mathbb{R}^2$

$$|f_1(x, y) - f_2(x, y)| \leq 3x^2,$$

therefore $f := f_1 - f_2$ is a d.c. function which is differentiable also at 0. Since

$$\frac{\partial f}{\partial y}(x, 0) = -2 \quad \text{for} \quad x \neq 0$$

and

$$\frac{\partial f}{\partial y}(0, 0) = 0,$$

the function $f$ is not strictly differentiable at $(0, 0)$.

REFERENCES


