

THEORY AND PRACTICE OF SYLVER COINAGE

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Abstract

Sylver Coinage is J. H. Conway's game of denominating coins. Two players take turns naming numbers that cannot be expressed as sums with non-negative multiples of numbers previously named. The player who chooses 1 loses. This paper presents a variety of general and specific results about Sylver Coinage.

“Look well to your change! Ring the silver on the four-pound weight!”
—Hawthorne, *The House of the Seven Gables*

1. Basic Knowledge

Sylver Coinage is J. H. Conway's game of denominating coins, familiar to readers of *Winning Ways* [1]. Two players take turns choosing numbers. Each number must not be a sum with multiples of numbers previously chosen. The player who chooses 1 loses.

Notation in Sylver Coinage varies from number-theoretic convention. The greatest common divisor of x_1, \dots, x_j is written $g(x_1, \dots, x_j)$, or g . If $g = 1$, the greatest number that cannot be expressed as a sum with multiples of members of $\{x_1, \dots, x_j\}$ is denoted $t(x_1, \dots, x_j)$, or t . If $S = \{x_1, \dots, x_j\}$, $g(S)$ denotes $g(x_1, \dots, x_j)$ and $t(S)$ denotes $t(x_1, \dots, x_j)$.

Sylver Coinage is named after J. J. Sylvester, who showed [2] that if $g(m, n) = 1$, then $t(m, n) = (m - 1)(n - 1) - 1$. For example, the greatest number that cannot be expressed as a sum of 5's and 6's is $(5 - 1)(6 - 1) - 1$, or 19.

Every game of Sylver Coinage must end. The value of g cannot increase as moves are added, and at most finitely many legal moves preserve the value of g . Thus every position has a well-defined Sprague-Grundy ordinal, which may be transfinite.

While Sprague-Grundy ordinals have been studied for some Sylver Coinage positions, most people care only who wins. Following Grundy and Smith [3], let a position be an \mathcal{N} -position if the player *next* to play can win; that is, if its Sprague-Grundy ordinal is nonzero. Otherwise let it be a \mathcal{P} -position (for *previous*).

A position S in Sylver Coinage is determined by a set of previous moves $\{x_1, \dots, x_j\}$. The expression Sn denotes the Sylver Coinage position obtained by adjoining n to the set S . This notation does not imply that n is a legal move in the position S , only that n is not an element of the set S .

It is often useful to multiply or divide an entire set of moves by some number. If $S = \{x_1, \dots, x_n\}$, then $S \times k = \{kx_1, \dots, kx_n\}$. Similarly, if $S = \{kx_1, \dots, kx_n\}$, then $S \div k = \{x_1, \dots, x_n\}$. Regardless of the value of $g(S)$, let $\bar{t}(S) = g(S)t(S \div g(S))$. For example, $\bar{t}(6, 8) = 10$, because $t(3, 4) = 5$. This extends the concept of t to positions with $g > 1$.

Different sets of moves may be equal as Sylver Coinage positions; for example, the position $\{4, 5, 10, 13\}$ equals the position $\{4, 5\}$. The moves 10 and 13 can be expressed as sums of 4's and 5's, so they do not affect play in the position. They are said to be *eliminated* by 4 and 5. A set of moves is a *canonical* representation of a position if it contains no superfluous moves. Any position can be made canonical by deleting the moves that are eliminated by the other moves. Mutual elimination is impossible: no number can help eliminate a smaller number!

Sylver Coinage positions have a natural partial order. If S and T are positions, write $T > S$ (T is *later* than S) if S and T are unequal and $T = Sx_1, \dots, x_j$ for some numbers x_1, \dots, x_j .

Every position partitions the set of numbers into legal and illegal moves. For example, in the position $\{4, 5\}$ the moves 1, 2, 3, 6, 7, and 11 are legal; all other numbers are illegal. In any position, all moves that lower the value of g are legal, so the set of legal moves is infinite if and only if $g > 1$.

In a position with $g = 1$, a legal move is called an *end* if it does not eliminate any other legal move. In particular, t is always an end. A position S where $t(S)$ is the only end is called an *ender*. By this definition, in an ender every legal move other than t eliminates t . Enders are important because they allow strategy stealing. Provided that $t > 1$, the player on the move in an ender must be able to win. Suppose he plays t . If the second player has a winning reply u , the first player can play u instead of t and reach the same winning position as the second player, because $Stu = Su$. Thus every ender with $t > 1$ is \mathcal{N} .

A *quiet ender* is an ender in which each legal move n less than t can eliminate t without multiples of n . For a counterexample let $S = \{4, 5, 7\}$. The legal moves in S are 1, 2, 3, and 6, so $t = 6$. Every legal move less than 6 eliminates 6, so S is an ender. But 3 does not *quietly* eliminate 6, because 6 cannot be expressed as a sum of a single 3 and multiple 4's, 5's, and

7's. Thus S is not a quiet ender. An ender that is not a quiet ender is an *unquiet ender*. Any position with $g = 1$ is thus a quiet ender, an unquiet ender, or a non-ender.

R. L. Hutchings has shown [1] that every position $\{m, n\}$ with $g = 1$ is an ender.¹ The only such position with $t = 1$ is $\{2, 3\}$, so every other position of the form $\{m, n\}$ is \mathcal{N} . It follows at once that if p is a prime greater than 3, $\{p\}$ is \mathcal{P} . This is Hutchings's *p-theorem* [1].

A powerful theorem in the analysis of Sylver Coinage is the *Quiet End Theorem* [1]: if z is coprime with each of x and y , then $(S \times x)z$ is a quiet ender if and only if $(S \times y)z$ is. In particular this theorem applies when S is itself a quiet ender and $x = 1$. Then when $z > t(S)$, z is illegal; that is, $Sz = S$, so Sz is a quiet ender. By the Quiet End Theorem, $(S \times y)z$ is also a quiet ender for values of z that are coprime with y and greater than t .

Now, quiet enders are enders, so those other than $\{2, 3\}$ are \mathcal{N} -positions. This can let us eliminate all but finitely many possible winning moves in a position with prime g . For example, let $S = \{2, 3\}$ and $y = 2$, so $S \times 2 = \{4, 6\}$. Sufficiently large moves in $\{2, 3\}$ produce quiet enders, so sufficiently large odd moves in $\{4, 6\}$ produce quiet enders. Thus none of $5, 7, 9, \dots$ can win in the position $\{4, 6\}$. Since the moves 2 and 3 lose to each other, it follows that $\{4, 6\}$ is \mathcal{P} .

A prime multiple of a quiet ender is said to be *short*, because in looking for a winning move in such a position, we need consider only finitely many moves. A prime multiple of an unquiet ender or non-ender is said to be *long*. By the Quiet End Theorem, sufficiently large moves in long positions produce positions that are not quiet enders. In fact, it is easy to show that sufficiently large moves produce positions that are not enders at all. Let $T = (S \times p)z$. It is routine to verify that for sufficiently large z , $t(T) = z(p - 1) + pt(S)$. There is some s that does not quietly eliminate $t(S)$ in S , so $t(S) - s$ cannot be expressed as a sum with multiples of members of S . But then ps cannot be expressed as a sum with multiples of members of $S \times p$. If z is large enough that $t(T) - ps > t(T)/2$, then $t(T) - ps$ cannot eliminate $t(T)$ in T .

2. Computing Sylver Coinage

From here on Guy's notation [4] for winning moves is used. The statement $\{x_1, \dots, x_j\}$ $[w_1, \dots, w_g]$ means that w_1, \dots, w_g are all the winning moves in the position $\{x_1, \dots, x_j\}$.

Any position with $g = 1$ has finitely many legal move sequences and can be analyzed exhaustively. While these computations are not particularly interesting, they raise some general questions, such as the asymptotic behavior of the unique winning move in $\{4, x\}$ for odd x .

¹The only known positions $\{m, n\}$ in which t wins are: $\{2, 5\}$; $\{4, 5\}$; $\{5, 6\}$; $\{5, 9\}$; $\{8, 15\}$; $\{13, 14\}$; $\{13, 21\}$. Are there infinitely many others? (And do they all involve Fibonacci numbers?)

Guy's "Twenty Questions" paper [4] is a list of such questions and others. Some early research in Sylver Coinage entailed finding all the winning moves in positions $\{m, n\}$ for small m and n ; see for example Nowakowski's paper [5]. As m and n increase, the computation grows rapidly. From the starting position $\{12, 17\}$ a total of 158793 positions can be reached, counting $\{12, 17\}$ itself.

Positions with $g > 1$ are more interesting because they have infinitely many legal moves. The case $g = 2$ is special because of the *Periodicity Theorem* [1]: if $g(S) = 2$, then the computation of the winning moves in Sx for odd x must eventually become periodic. The proof shows that the analysis can be computed by a finite-state automaton because the number of potential winning moves in Sx is bounded over x . For example, let $S = \{8, 10, 22\}$; then $g(S) = 2$ and $t(S) = 14$. In analyzing an odd move n we need consider no odd replies greater than $n + 14$ (because n eliminates them) and no odd replies less than $n - 14$ (because they eliminate n). It suffices to analyze the odd replies $n - 14, n - 12, \dots, n - 2, n + 2, \dots, n + 12, n + 14$, along with the finitely many even replies.

The Periodicity Theorem implies that a finite computation suffices to determine whether a position with $g = 2$ has any winning moves.

A short position S with $g = 2$ can have few possible winning moves for the next player, for the Quiet End Theorem assures us that S has no odd winning move greater than $t(S \div 2)$. Thus it is not unusual to find short positions that are \mathcal{P} ; for example $\{4, 6\}$, $\{8, 14\}$, and $\{10, 16, 24\}$. On the other hand, in long positions with $g = 2$ the only known limit on possible winning moves is imposed by the Periodicity Theorem—usually a very high limit! For this reason, most long positions with $g = 2$ are \mathcal{N} , and their winning moves may be quite large. A few long positions with $g = 2$ are known to be \mathcal{P} , notably $\{8, 10, 22\}$ and the members of the series $\{8, 12, 8n + 2, 8n + 6\}$ for $n = 1, 2, 3, \dots$. (From this it can be shown that $\{8, 12\}$ is also \mathcal{P} .) Computationally speaking, a position with $g = 2$ may be called *Probably \mathcal{P}* if it has no odd winning moves and every even move either is known to lose or produces a long position.

While a finite analysis suffices to find any winning moves in a long position with $g = 2$, the computation may be very long. The unique winning move in $\{8, 30, 34\}$ is 49337. In $\{6, 44, 82\}$ it is obvious that 4 wins, reducing the position to $\{4, 6\}$; it is less obvious that 5993171 wins. The position $\{6, 40, 74\}$, though long, has no odd winning move; a period of length 403200 sets in at $x = 381091$. The long positions $\{16, 22, 24, 26, 28, 30, 34\}$ and $\{20, 22, 24, 26, 28, 36\}$ also have no odd winning moves, and are in fact \mathcal{P} . The long position $\{6, 50, 94\}$ has neither period nor odd winning move below 10^8 .

When $g = 3$, short positions are still often \mathcal{P} , but long ones are generally intractable. The number of legal moves in Sx is $o(2x)$, so, intuitively, the chance of finding no winning reply to x decreases as x increases. For example, in $\{12, 15, 21\}$, the move 100 has 63 possible

Proof. Both of $n, t-n$ cannot be illegal, for then their sum would be illegal; but t is by definition the greatest *legal* move in S . Now suppose that n is a legal move in S and is less than t . By the definition of a quiet ender, t is the sum of n once and elements of S with multiples allowed. Thus $t-n$ can be expressed as a sum with multiples of elements of S . In other words, $t-n$ is an illegal move. \square

Since the proof works in reverse, the Strong Antisymmetry Principle exactly characterizes quiet enders.

Proposition. *If S is a quiet ender, then $t(S)$ is odd.*

Proof. If t is even, the Antisymmetry Principle implies that $t/2$ is both legal and illegal! \square

It turns out that unquiet enders are much like quiet ones.

Proposition. *Let S be an ender. Then for every n such that $0 < n < t$ and $n \neq t/2$, exactly one of $n, t-n$ is a legal move in S .*

Proof. We have seen that n and $t-n$ cannot both be illegal. Suppose without loss of generality that $n > t/2$. By the definition of an ender, t is the sum with multiples of n and elements of S . Since $n > t/2$, n must occur only once in this sum. Thus $t-n$ can be expressed as a sum with multiples of elements of S . In other words, $t-n$ is an illegal move. \square

The proof of this *Weak Antisymmetry Principle* looks very like that of the Strong Antisymmetry Principle. The only difference is that antisymmetry may fail at $t/2$ for unquiet enders.

Proposition. *Let S be an ender and $t(S)$ be odd. Then S is a quiet ender.*

Proof. When $t/2$ is not a number, the Weak Antisymmetry Principle reduces to the Strong Antisymmetry Principle. A position that satisfies the Strong Antisymmetry Principle is a quiet ender. \square

Combining these results gives a neat characterization of quietness in enders, the *Odds and Enders Theorem*:

Theorem. *Let S be an ender. Then S is a quiet ender if and only if $t(S)$ is odd.*

We can now prove a stronger form of Hutchings's basic theorem.

Proposition. *If m and n are coprime, $\{m, n\}$ is a quiet ender.*

Proof. Hutchings has shown that the position is an ender, and Sylvester has shown that $t(m, n) = (m - 1)(n - 1) - 1$. Since m and n are coprime, at least one of them must be odd. It follows that t is odd, so the position is a quiet ender. \square

The Antisymmetry Principles suggest an infallible method of generating enders:

Definition. *Let S be a position with $g = 1$. The enclosure of S , written $E(S)$, is the position obtained by adjoining to S every legal move n such that $t/2 < n < t$ and $t - n$ is a legal move.*

For example, let $S = \{7, 11, 13, 15\}$. Then $t(S) = 23$ and the legal moves in S are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 17, 19 and 23. The pairs of legal moves that sum to 23 are 4+19 and 6+17. Adjoining 19 and 17 to S gives $E(S) = \{7, 11, 13, 15, 17, 19\}$, which is an ender.

Thus the enclosure of a position is obtained by attempting to impose antisymmetry on it. But does this always produce an ender?

Proposition. *Let S be a position with $g = 1$. Then $t(E(S)) = t(S)$.*

Proof. Suppose that $t(E(S)) < t(S)$. Then $t(S)$ can be expressed as a sum with multiples of elements of $E(S)$. Such a sum must include one or more elements of $E(S) - S$, since $t(S)$ cannot be expressed as a sum with multiples of elements of S . The numbers adjoined to S to form $E(S)$ are all greater than $t(S)/2$, so the sum can include at most one of them. Call it n , and let $m = t(S) - n$. Then m is also a sum with multiples of elements of $E(S)$. By the definition of E , both n and m are legal in S , and $m < t(S)/2$. Being legal in S , m cannot be expressed as a sum with multiples of members of S . But all the adjoined members in $E(S)$ are greater than $t(S)/2$ and thus greater than m , so they cannot be included in a sum to m . This shows that m cannot be expressed as a sum with multiples of members of $E(S)$, and we have a contradiction. \square

Theorem. *Let S be a position with $g = 1$. Then $E(S)$ is an ender.*

Proof. This now follows from the definition of E . For $t(E(S)) = t(S)$, and every possible violation of weak antisymmetry with respect to t is removed by the process of enclosure. \square

Obviously when S is an ender, $E(S) = S$. For arbitrary S enclosure satisfies a minimality condition:

Proposition. *Let S be a position with $g = 1$. If $S \leq T \leq E(S)$ and T is an ender, then $T = E(S)$.*

Proof. Since the t operator is monotone, $t(S) \geq t(T) \geq t(E(S))$. We have shown that $t(E(S)) = t(S)$, so $t(T) = t(S)$. Suppose that T is a proper subposition of $E(S)$, so that for some n , n is legal in T and illegal in $E(S)$. Since $S \leq T$, n is legal in S . Thus n is one of the numbers adjoined to S to produce $E(S)$. By definition of $E(S)$, $m = t - n$ is also a legal move in S , and $m < t/2$. Since m is also legal in $E(S)$, it must be legal in T . Thus T has two distinct legal moves m and n that sum to t , violating the Weak Antisymmetry Principle. Such a position cannot be an ender. \square

Like t , the function E can be extended to a function over all positions: $\bar{E}(S) = E(S \div g(S)) \times g(S)$. For example, $\bar{E}(14, 22, 26, 30) = \{14, 22, 26, 30, 34, 38\}$.

4. Arithmetic Progressions

This remarkable theorem is due to J. B. Roberts [6]. Let a_0, \dots, a_s be an increasing arithmetic progression with $a_0 \geq 2$ and successive difference d , and suppose that $g(a_0, \dots, a_s) = 1$. Then

$$t(a_0, \dots, a_s) = \left(\left\lfloor \frac{a_0 - 2}{s} \right\rfloor + 1 \right) \cdot a_0 + (d - 1)(a_0 - 1) - 1.$$

When $s = 1$ this reduces to Sylvester's Formula. Like Sylvester's Formula, Roberts's Formula has an analogue in Sylver Coinage:

Theorem. *Let $S = \{a_0, \dots, a_s\}$ satisfy the conditions of Roberts's Theorem. Assume also that $s \leq a_0 - 1$. Then*

- A) *if $s = 1$, S is a quiet ender;*
- B) *if $a_0 = 3$ and $s = 2$, S is an unquiet ender;*
- C) *if $a_0 > 3$, S is a quiet ender if $s \mid a_0 - 2$ and not an ender otherwise.*

Proof. I only sketch the proof, omitting demonstrations of the values of t and the moves that eliminate them.

In case A the set has only two members and is covered by Hutchings's Theorem.

In case B, $S = \{3, 3 + d, 3 + 2d\}$, and the value of t is $2d$, which is even, so S is not a quiet ender. The legal moves are just the positive values of $d - 3k$ and $2d - 3k$ for $k \geq 0$. By congruence modulo 3, the only two such moves that can sum to $2d$ are d and d , so weak antisymmetry holds.

Case C has two subcases.

I. If $s = a_0 - 1$, then clearly s does not divide $a_0 - 2$. In this subcase, $t = sd$, and the moves that fail to eliminate t are the values of kd where $1 \leq k \leq s$ and k does not divide s . In particular $s - 1$ does not divide s , so S is a non-ender.

II. If $s < a_0 - 1$, then the moves that fail to eliminate t are $t - kd$ for $1 \leq k \leq r$, where r is the remainder on dividing $a_0 - 2$ by s . If $r = 0$ then S is an ender; otherwise it is not. Now if $r = 0$, then

- i) if a_0 is odd, $a_0 - 2$ is also odd; since this is divisible by s , $(a_0 - 2)/s$ is odd, and the first term in Roberts's Formula is even;
- ii) if a_0 is even, the first term in Roberts's Formula is even because it contains a_0 as a factor.

In both cases the second term $(d - 1)(a_0 - 1)$ is even because d and a_0 are coprime and cannot both be even. Thus t is odd and S is a quiet ender. □

5. Synthetic Methods

A natural problem in Sylver Coinage is: given a set of numbers, is there a position in which those numbers are winning moves? If so, is there a position in which those numbers are the only winning moves?

For a single number the answer to both questions is yes. This is the *Separation Lemma*:

Lemma. *Suppose that x and y both win in position S . Then there is a position T such that $T > S$, x wins in T , and y does not.*

Proof. We are given that Sx and Sy are \mathcal{P} -positions. First, $Sxy \neq Sx$; that is, x cannot eliminate y . If it did, then x would be a winning move in Sy . Therefore Sxy is an \mathcal{N} -position; let z be a winning move in it, and let $T = Syz$. Since y is not a legal move in T , it is not a winning move. But x is a legal move in T , for otherwise $Sxyz = Syz$, whence Syz is \mathcal{P} . But z is a legal move in the \mathcal{P} -position Sy , so Syz must be \mathcal{N} . □

This suffices to prove the *Single Win Theorem*:

Theorem. *Let n be a number greater than 1. Then there is a position in which n alone wins.*

Proof. We know that 2 alone wins in $\{3\}$, and 3 alone wins in $\{2\}$. For any n greater than 3,

let p be a prime greater than 3 such that $p \neq n$. By Hutchings's Theorem, $\{n, p\}$ is \mathcal{N} , so it has a winning move r . Then $\{n, p, r\}$ is \mathcal{P} . Let $S_0 = \{p, r\}$. Then $S_0 n \neq S_0$, for $S_0 n$ is \mathcal{P} while $S_0 = \{p, r\}$ is \mathcal{N} by Hutchings's Theorem. Thus n is a legal and winning move in S_0 .

If n is the only winning move in S_0 , we are done. Suppose that S_0 has some other winning move n' . By the Separation Lemma there exists $S_1 > S_0$ in which n wins and n' does not.

If n is the only winning move in S_1 , we are done. If not, repeat the process. Eventually it must terminate, since there cannot be an infinite series of positions

$$S_0 < S_1 < S_2 < \dots$$

When it terminates in some S_j , n alone wins in S_j . □

To illustrate, let $n = 14$. Using $p = 5$ gives $\{5, 14\}$ [18]. Solving $S_0 = \{5, 18\}$ gives $[14, 16, 17]$. Let us eliminate 16 by solving $\{5, 14, 16, 18\}$; the unique winning move is 17. Then $S_1 = \{5, 16, 17, 18\}$, and this has the desired unique winning move 14.

Conjecture. *Any set of mutually indivisible numbers greater than 3 is the set of winning moves in some position.*

Here is an example of an attempt at constructing such a position. Let $[4, 5, 6, 7]$ be the desired set of winning moves. Start with any number bigger than all of them and indivisible by them; 9 will do. Solve $\{4, 9\}$ to get 19, so 4 wins in $\{9, 19\}$. Does 5 win? No: $\{5, 9, 19\}$ [31]. However, $\{4, 9, 19, 31\} = \{4, 9, 19\}$, so we can add 31 to the target position. Now we have $\{9, 19, 31\}$ [4, 5, ...].

Does 6 win in $\{9, 19, 31\}$? No: $\{6, 9, 19, 31\}$ [17, 20, 22]. But adjoining 20 to the position has no effect on 4 or 5. We now have $\{9, 19, 20, 31\}$ [4, 5, 6, ...].

Does 7 win in $\{9, 19, 20, 31\}$? No: $\{9, 19, 20, 31\}$ [15]. Adjoining 15 to the position has no effect on 5 ($15 = 5 + 5 + 5$) or 6 ($15 = 6 + 9$), but 15 is not eliminated by 4.

Instead of adjoining 15, can we adjoin *two* moves that do not affect 4, 5, and 6? No; the only such move that does affect 7 is 24.

When we reach a dead end, the thing to do is backtrack. At any stage of the search we can try adjoining several moves instead of one. As a last resort we can choose a new starting move; there are infinitely many to choose from!

Probabilistically, the search is certain to succeed—unless it can somehow be proven to be certain to fail. As for $[4, 5, 6, 7]$, one position that works is $\{17, 18, 27, 33, 43\}$. By good fortune,

adjoining 56 to this position lets us add 9 to the set of winning moves: $\{17, 18, 27, 33, 43, 56\}$ [4, 5, 6, 7, 9].

A more speculative conjecture is:

Conjecture. *For any $n \geq 3$ there is a position with $g = 2$ in which n wins.*

Other values of g produce other conjectures.

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